MONOMORPHISM OPERATOR AND PERPENDICULAR OPERATOR

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Abstract. For a quiver Q, a k-algebra A, and a full subcategory $\mathcal{X}$ of A-mod, the monomorphism category $\mathrm{Mon}(Q,\mathcal{X})$ is introduced. The main result says that if T is an A-module such that there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to D(A_A) \to 0$ with each $T_i \in \mathrm{add}(T)$, then $\mathrm{Mon}(Q, {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T)$; and if T is cotilting, then $kQ \otimes_k T$ is a unique cotilting Λ-module, up to multiplicities of indecomposable direct summands, such that $\mathrm{Mon}(Q, {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T)$.

As applications, the category of the Gorenstein-projective $(kQ \otimes_k A)$-modules is characterized as $\mathrm{Mon}(Q, \mathcal{GP}(A))$ if A is Gorenstein; the contravariantly finiteness of $\mathrm{Mon}(Q,\mathcal{X})$ can be described; and a sufficient and necessary condition for $\mathrm{Mon}(Q, A)$ being of finite type is given.

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1. Introduction

1.1. With a quiver Q and a k-algebra A, one can associate the monomorphism category $\mathrm{Mon}(Q, A)$ ([LZ]). If $Q = \bullet \to \bullet$ it is called *the submodule category* and denoted by $\mathcal{S}(A)$. If $Q = n\bullet \to \cdots \to \bullet 1$ it is called *the filtered chain category* in D. Simson [S]; and it is denoted by $\mathcal{S}_n(A)$ in [Z].

G. Birkhoff [B] initiates the study of $\mathcal{S}(\mathbb{Z}/\langle p^t\rangle)$. C. M. Ringel and M. Schmidmeier ([RS1] - [RS3]) have extensively studied $\mathcal{S}(A)$. In particular, the Auslander-Reiten theory of $\mathcal{S}(A)$ is explicitly given ([RS2]). Since then the monomorphism category receives more attention. In [Z] relations among $\mathcal{S}_n(A)$ and the Gorenstein-projective modules and cotilting theory are given. D. Kussin, H. Lenzing, and H. Meltzer [KLM1] establish a surprising link between the stable submodule category and the singularity theory via weighted projective lines (see also [KLM2]). In [XZZ] the Auslander-Reiten theory of $\mathcal{S}(A)$ is extended to $\mathcal{S}_n(A)$. For more related works we refer to [A], [RW], [SW], [Mo], [C1], [C2], and [RZ].

1.2. Let $\mathcal{X}$ be a full subcategory of A-mod. We also define the monomorphism category $\mathrm{Mon}(Q,\mathcal{X})$. For an A-module T, let ${}^{\perp}T$ be the full subcategory of A-mod consisting of those modules X with $\mathrm{Ext}^i_A(X,T) = 0, \ \forall\ i \geq 1$. The main result of this paper gives a reciprocity of the monomorphism operator $\mathrm{Mon}(Q,-)$ and the left perpendicular operator ${}^{\perp}$. Namely, if T is an A-module such

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that there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to D(A_A) \to 0$ with each $T_i \in \mathrm{add}(T)$, then $\mathrm{Mon}(Q, {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T)$ (Theorem 3.1); and if T is a cotilting A-module, then $kQ \otimes_k T$ is a unique cotilting Λ-module, up to multiplicities of indecomposable direct summands, such that $\mathrm{Mon}(Q, {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T)$ (Theorem 4.1).

Theorems 3.1 and 4.1 generalize [Z, Theorem 3.1(i) and (ii)] for $Q = \bullet \to \cdots \to \bullet$. However, the arguments in [Z] can not be generalized to the general case (cf. 3.1 and 4.1 below). Here we adopt new treatments, in particular by using an adjoint pair $(\mathrm{Coker}_i, S(i) \otimes -)$ and Lemma 4.4.

1.3. Our main results have some applications, which generalize the corresponding results in [Z].

The category $\mathcal{GP}(A)$ of the Gorenstein-projective A-modules is Frobenius (cf. [AB], [AR], [EJ]), and hence the corresponding stable category is triangulated ([H]). If A is Gorenstein (i.e., $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$), then $\mathcal{GP}(A) = {}^{\perp}A$ ([EJ, Corollary 11.5.3]). Taking $T = {}_AA$ in Theorem 3.1 we have $\mathcal{GP}(\Lambda) = \mathrm{Mon}(Q, \mathcal{GP}(A))$ if A is Gorenstein.

M. Auslander and I. Reiten [AR, Theorem 5.5(a)] have established a deep relation between resolving contravariantly finite subcategories and cotilting theory, by asserting that $\mathcal{X}$ is resolving and contravariantly finite with $\widehat{\mathcal{X}} = A$-mod if and only if $\mathcal{X} = {}^{\perp}T$ for some cotilting A-module T, where $\widehat{\mathcal{X}}$ is the full subcategory of A-mod consisting of those modules X, such that there is an exact sequence $0 \to X_m \to \cdots \to X_0 \to X \to 0$ with each $X_i \in \mathcal{X}$. It is natural to ask when is $\mathrm{Mon}(Q, \mathcal{X})$ contravariantly finite in Λ-mod? As an application of Theorem 4.1 and [AR, Theorem 5.5(a)], we see that $\mathrm{Mon(Q}, \mathcal{X})$ is resolving and contravariantly finite with $\widehat{\mathrm{Mon}(Q, \mathcal{X})} = \Lambda$-mod if and only if $\mathcal{X}$ is resolving and contravariantly finite with $\widehat{\mathcal{X}} = A$-mod (Theorem 5.1).

It is well-known that the representation type of $\mathrm{Mon}(Q, A)$ is different from the ones of A and of $\Lambda = kQ \otimes_k A$. For example, $k[x]/\langle x^t \rangle$ is of finite type, while $k(\bullet \to \bullet) \otimes_k k[x]/\langle x^t \rangle$ is of finite type if and only if $t \le 3$, and $\mathcal{S}_2(k[x]/\langle x^t \rangle)$ is of finite type if and only if $t \le 5$. If $t > 6$ then $\mathcal{S}_2(k[x]/\langle x^t \rangle)$ is of "wild" type, while $\mathcal{S}_2(k[x]/\langle x^6 \rangle)$ is of "tame" type ([S], Theorems 5.2 and 5.5). A complete classification of indecomposable objects of $\mathcal{S}_2(k[x]/\langle x^6 \rangle)$ is exhibited in [RS3]. Inspired by Auslander's classical result: A is representation-finite if and only if there is an A-generator-cogenerator M such that $\mathrm{gl.dim}\,\mathrm{End}_A(M) \le 2$ ([Au], Chapter III), by using Theorem 4.1 we prove that $\mathrm{Mon}(Q, A)$ is of finite type if and only if there is a generator and relative cogenerator M of $\mathrm{Mon}(Q, A)$ such that $\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \le 2$ (Theorem 6.1).

2. **Preliminaries on monomorphism categories**

In this section we fix notations, and give necessary definitions and facts.

2.1. Throughout this paper, k is a field, Q is a finite acyclic quiver (i.e., a finite quiver without oriented cycles), and A is a finite-dimensional k-algebra. Denote by kQ the path algebra of Q over k. Put $\Lambda = kQ \otimes_k A$, and $D = \mathrm{Hom}_k(-, k)$. Let $P(i)$ (resp. $I(i)$) be the indecomposable projective (resp. injective) kQ-module, and $S(i)$ the simple kQ-module, at $i \in Q_0$. By A-mod we denote the category of finite-dimensional left A-modules. For an A-module T, let $\mathrm{add}(T)$ be the full the subcategory of A-mod consisting of all the direct sums of indecomposable direct summands of T.

2.2. Given a finite acyclic quiver $Q = (Q_0, Q_1, s, e)$ with Q_0 the set of vertices and Q_1 the set of arrows, we write the conjunction of a path p of Q from right to left, and let $s(p)$ and $e(p)$ be respectively the starting and the ending point of p. The notion of representations of Q over k can be extended as follows. By definition ([LZ]), *a representation X of Q over A* is a datum $X = (X_i,\ X_\alpha,\ i \in Q_0,\ \alpha \in Q_1)$, or simply $X = (X_i, X_\alpha)$, where each X_i is an A-module, and each $X_\alpha : X_{s(\alpha)} \to X_{e(\alpha)}$ is an A-map. It is *a finite-dimensional representation* if so is each X_i. We call X_i *the i-th branch* of X. A morphism f from X to Y is a datum $(f_i,\ i \in Q_0)$, where $f_i : X_i \to Y_i$ is an A-map for $i \in Q_0$, such that for each arrow $\alpha : j \to i$ the following diagram

$$\begin{tikzcd} X_j \arrow[r, "f_j"] \arrow[d, "X_\alpha"'] & Y_j \arrow[d, "Y_\alpha"] \\ X_i \arrow[r, "f_i"] & Y_i \end{tikzcd} \tag{2.1}$$

commutes. Denote by $\mathrm{Rep}(Q, A)$ the category of finite-dimensional representations of Q over A. Note that a sequence of morphisms $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathrm{Rep}(Q, A)$ is exact if and only if $0 \longrightarrow X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \longrightarrow 0$ is exact in A-mod for each $i \in Q_0$.

Lemma 2.1. ([LZ, Lemma 2.1]) *We have an equivalence* Λ-mod $\cong \mathrm{Rep}(Q, A)$ *of categories.*

In the following we will identify a Λ-module with a representation of Q over A. If $T \in A$-mod and $M \in kQ$-mod with $M = (M_i, i \in Q_0, M_\alpha, \alpha \in Q_1) \in \mathrm{Rep}(Q, k)$, then $M \otimes_k T \in \Lambda$-mod with $M \otimes_k T = (M_i \otimes_k T = T^{\dim_k M_i},\ i \in Q_0,\ M_\alpha \otimes_k \mathrm{Id}_T,\ \alpha \in Q_1) \in \mathrm{Rep}(Q, A)$.

2.3. Here is the central notion of this paper.

Definition 2.2. (i) ([LZ]) *A representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1) \in \mathrm{Rep}(Q, A)$ is a monic representation of Q over A, or a monic Λ-module, if $\delta_i(X)$ is an injective A-map for each $i \in Q_0$, where*

$$\delta_i(X) = (X_\alpha)_{\alpha \in Q_1,\ e(\alpha) = i} : \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha) = i}} X_{s(\alpha)} \longrightarrow X_i.$$

Denote by $\mathrm{Mon}(Q, A)$ *the full subcategory of* $\mathrm{Rep}(Q, A)$ *consisting of all the monic representations of Q over A, which is called the monomorphism category of A over Q.*

(ii) *Let $\mathcal{X}$ be a full subcategory of A-mod. Denote by* $\mathrm{Mon}(Q, \mathcal{X})$ *the full subcategory of* $\mathrm{Mon}(Q, A)$ *consisting of all the monic representations $X = (X_i, X_\alpha)$, such that $X_i \in \mathcal{X}$ and* $\mathrm{Coker}\,\delta_i(X) \in \mathcal{X}$ *for all $i \in Q_0$. We call* $\mathrm{Mon}(Q, \mathcal{X})$ *the monomorphism category of $\mathcal{X}$ over Q.*

If $\mathcal{X} = A$-mod then $\mathrm{Mon}(Q, \mathcal{X}) = \mathrm{Mon}(Q, A)$. For $M \in kQ$-mod and $T \in A$-mod, it is clear that if $M \in \mathrm{Mon}(Q, k)$ then $M \otimes_k T \in \mathrm{Mon}(Q, A)$. In particular, $P(i) \otimes_k T \in \mathrm{Mon}(Q, A)$ for each $i \in Q_0$.

Note that $D(\Lambda_\Lambda) \cong D(kQ_{kQ}) \otimes_k D(A_A)$ as left Λ-modules. We need the following fact.

Lemma 2.3. ([LZ, Proposition 2.4]) *Let* $\mathrm{Ind}\mathcal{P}(A)$ *(resp.* $\mathrm{Ind}\mathcal{I}(A)$*) denote the set of pairwise non-isomorphic indecomposable projective (resp. injective) A-modules. Then*

$$\mathrm{Ind}\mathcal{P}(\Lambda) = \{P(i) \otimes_k P \mid i \in Q_0,\ P \in \mathrm{Ind}\mathcal{P}(A)\} \subseteq \mathrm{Mon}(Q, A),$$

and

$$\mathrm{Ind}\mathcal{I}(\Lambda) = \{I(i) \otimes_k I \mid i \in Q_0,\ I \in \mathrm{Ind}\mathcal{I}(A)\}.$$

In particular, for $M \in kQ$-mod we have $\mathrm{proj.dim}(M \otimes_k A) \leq 1$, *and* $\mathrm{inj.dim}(M \otimes_k D(A_A)) \leq 1$.

2.4. Given $X = (X_j, X_\alpha) \in \Lambda$-mod, for each $i \in Q_0$ we have functors F_i and F_i^+ from Λ-mod to A-mod, respectively induced by $F_i(X) = X_i$ and $F_i^+(X) := \bigoplus\limits_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)}$ (if i is a source then $F_i^+(X) := 0$).

We write $\mathrm{Coker}\,\delta_i(X)$ (cf. Definition 2.2 (i)) as $\mathrm{Coker}_i(X)$. Then we have a functor Coker_i : Λ-mod $\longrightarrow$ A-mod, explicitly given by $\mathrm{Coker}_i(X) := X_i / \sum\limits_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \mathrm{Im}\, X_\alpha$ (if i is a source then $\mathrm{Coker}_i(X) := X_i$). So we have an exact sequence of functors $F_i^+ \stackrel{\delta_i}{\longrightarrow} F_i \stackrel{\pi_i}{\longrightarrow} \mathrm{Coker}_i \longrightarrow 0$, i.e., we have the exact sequence of A-modules

$$\bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \stackrel{\delta_i(X)}{\longrightarrow} X_i \stackrel{\pi_i(X)}{\longrightarrow} \mathrm{Coker}_i(X) \longrightarrow 0$$

for each $X \in \Lambda$-mod, where $\pi_i(X)$ is the canonical map. It is clear that F_i^+ and F_i are exact, and Coker_i is right exact (by Snake Lemma). For $i, j \in Q_0$ and $T \in A$-mod, we have

$$\mathrm{Coker}_i(P(j) \otimes_k T) = \begin{cases} T, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases} \tag{2.2}$$

Lemma 2.4. *For each $i \in Q_0$, the restriction of functor* Coker_i *to* $\mathrm{Mon}(Q, A)$ *is exact.*

Proof. Let $0 \to (X_i, X_\alpha) \to (Y_i, Y_\alpha) \to (Z_i, Z_\alpha) \to 0$ be an exact sequence in $\mathrm{Mon}(Q, A)$. Then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus X_{s(\alpha)} \arrow[r] \arrow[d, "\delta_i(X)"] & \bigoplus Y_{s(\alpha)} \arrow[r] \arrow[d, "\delta_i(Y)"] & \bigoplus Z_{s(\alpha)} \arrow[r] \arrow[d, "\delta_i(Z)"] & 0 \\
0 \arrow[r] & X_i \arrow[r] & Y_i \arrow[r] & Z_i \arrow[r] & 0.
\end{tikzcd}$$

Then the assertion follows from Snake Lemma since $\delta_i(Z)$ is injective. ■

Recall from [AR] that $\mathcal{X}$ is *resolving* if $\mathcal{X}$ contains all the projective A-modules, $\mathcal{X}$ is closed under taking extensions, kernels of epimorphisms, and direct summands. Dually one has *a coresolving subcategory.*

Lemma 2.5. *Let $\mathcal{X}$ be a full subcategory of A-mod. Then*

(i) $\mathrm{Mon}(Q,\mathcal{X})$ *is closed under taking extensions (resp. kernels of epimorphisms, direct summands) if and only if* $\mathcal{X}$ *is closed under taking extensions (resp. kernels of epimorphisms, direct summands).*

(ii) $\mathrm{Mon}(Q,\mathcal{X})$ *is resolving if and only if* $\mathcal{X}$ *is resolving. In particular,* $\mathrm{Mon}(Q,A)$ *is resolving.*

Proof. (i) can be similarly proved as Lemma 2.4. For (ii), by Lemma 2.3 the branches of projective Λ-modules are projective A-modules. From this and (i) the assertion follows. ■

3. **Reciprocity**

3.1. This section is to prove the following reciprocity of the monomorphism operator and the left perpendicular operator.

Theorem 3.1. *Let T be an A-module such that there is an exact sequence* $0\to T_m\to\cdots\to T_0\to D(A_A)\to 0$ *with each* $T_j\in\mathrm{add}(T)$*, then* $\mathrm{Mon}(Q,\ {}^{\perp}T)=\ {}^{\perp}(kQ\otimes_k T)$.

For $Q=\bullet\to\cdots\to\bullet$ this result has been obtained in [Z, Theorem 3.1(i)]. Since some adjoint pairs in [Z, Lemma 1.2] are not available here, the arguments in [Z] can not be generalized to the general case. Here we adopt the following adjoint pair $(\mathrm{Coker}_i, S(i)\otimes -)$.

3.2. The following observation will be used throughout this section.

Lemma 3.2. *Let* $X=(X_i,X_\alpha)\in\Lambda$*-mod and* $T\in A$*-mod. Then for each* $i\in Q_0$ *we have an isomorphism of abelian groups which is natural in both positions*

$$\mathrm{Hom}_A(\mathrm{Coker}_i(X),T)\cong \mathrm{Hom}_\Lambda(X,S(i)\otimes_k T).$$

Proof. If we write $S(i)\otimes_k T\in\Lambda$-mod as (Y_j,Y_α), then $Y_j=0$ for $j\neq i$ and $Y_i=T$. Consider the homomorphism $\Psi:\mathrm{Hom}_A(\mathrm{Coker}_i(X),T)\to\mathrm{Hom}_\Lambda(X,S(i)\otimes_k T)$ given by

$$f\mapsto \Psi(f)=(g_j, j\in Q_0):\ X\to S(i)\otimes_k T,\ \forall f\in\mathrm{Hom}_A(\mathrm{Coker}_i(X),T),$$

where $g_j=0$ for $j\neq i$, and $g_i=f\ \pi_i(X):X_i\to T$ with the canonical map $\pi_i(X):X_i\to\mathrm{Coker}_i(X)$. By (2.1) it is clear that $\Psi(f)\in\mathrm{Hom}_\Lambda(X,S(i)\otimes_k T)$ and Ψ is surjective. It is injective since $\pi_i(X)$ is surjective. ■

3.3. We need the following fact.

Lemma 3.3. *Let T be an A-module. For each* $i\in Q_0$ *we have* ${}^{\perp}(kQ\otimes_k T)=\ {}^{\perp}(\bigoplus\limits_{i\in Q_0}(S(i)\otimes_k T))$.

Proof. Put $S=\bigoplus\limits_{i\in Q_0}S(i)$, and J to be the Jacobson radical of kQ with $J^l=0$. Let $X\in{}^{\perp}(kQ\otimes T)$. By the exact sequence $0\to J\otimes_k T\to kQ\otimes_k T\to S\otimes_k T\to 0$ we get the exact sequence:

$$\cdots\to\mathrm{Ext}^j_\Lambda(X,kQ\otimes_k T)\to\mathrm{Ext}^j_\Lambda(X,S\otimes_k T)\to\mathrm{Ext}^{j+1}_\Lambda(X,J\otimes_k T)\to\cdots.$$

Since kQ is hereditary, $J\in\mathrm{add}(kQ)$ and hence $\mathrm{Ext}^j_\Lambda(X,J\otimes_k T)=0$, $\forall\, j\geq 1$. Thus $X\in{}^{\perp}(S\otimes_k T)$.

Conversely, let $X \in {}^{\perp}(S \otimes_k T)$. From the exact sequence $0 \to J^{l-1} \otimes_k T \to J^{l-2} \otimes_k T \to (J^{l-2}/J^{l-1}) \otimes_k T \to 0$ and by $J^{l-1} \otimes_k T,\ J^{l-2}/J^{l-1} \otimes_k T \in \mathrm{add}(S \otimes_k T)$ we see $X \in {}^{\perp}(J^{l-2} \otimes_k T)$. Continuing this process we finally see $X \in {}^{\perp}(J^0 \otimes_k T) = {}^{\perp}(kQ \otimes_k T)$. ■

Proposition 3.4. *We have* $\mathrm{Mon}(Q, A) = {}^{\perp}(kQ \otimes_k D(A_A))$.

Proof. By Lemma 3.3 it suffices to prove the following equality, for each $i \in Q_0$:

$$^{\perp}(S(i) \otimes_k D(A_A)) = \{X = (X_j, X_\alpha) \in \mathrm{Rep}(Q, A) \mid \delta_i(X) \text{ is injective}\}.$$

Let P_X be the projective cover of X. Applying functor F_i^+ and F_i to the exact sequence $0 \to \Omega(X) \to P_X \to X \to 0$ we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_i^+(\Omega(X)) & \longrightarrow & F_i^+(P_X) & \longrightarrow & F_i^+(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_i(\Omega(X))} & & \downarrow{\scriptstyle \delta_i(P_X)} & & \downarrow{\scriptstyle \delta_i(X)} & & \\ 0 & \longrightarrow & F_i(\Omega(X)) & \longrightarrow & F_i(P_X) & \longrightarrow & F_i(X) & \longrightarrow & 0. \end{array}$$

By Snake Lemma we have the exact sequence

$$0 \to \mathrm{Ker}\, \delta_i(X) \to \mathrm{Coker}_i(\Omega(X)) \to \mathrm{Coker}_i(P_X) \to \mathrm{Coker}_i(X) \to 0. \qquad (*)$$

Assume that $\delta_i(X)$ is injective. Applying $\mathrm{Hom}_A(-, D(A_A))$ to $(*)$ and by Lemma 3.2 we get the following exact sequence (with Hom omitted)

$$0 \to (X, S(i) \otimes_k D(A_A)) \to (P_X, S(i) \otimes_k D(A_A)) \to (\Omega(X), S(i) \otimes_k D(A_A)) \to 0. \qquad (**)$$

Applying $\mathrm{Hom}_\Lambda(-, S(i) \otimes_k D(A_A))$ to $0 \to \Omega(X) \to P_X \to X \to 0$ we get the exact sequence

$$0 \to (X, S(i) \otimes D(A_A)) \to (P_X, S(i) \otimes D(A)) \to (\Omega(X), S(i) \otimes D(A)) \to \mathrm{Ext}^1_\Lambda(X, S(i) \otimes D(A)) \to 0.$$

Comparing it with $(**)$ we see $\mathrm{Ext}^1_\Lambda(X, S(i) \otimes_k D(A_A)) = 0$. By Lemma 2.3 we have $\mathrm{inj.dim}(S(i) \otimes_k D(A_A)) \leq 1$, so $X \in {}^{\perp}(S(i) \otimes_k D(A_A))$.

Conversely, assume $X \in {}^{\perp}(S(i) \otimes_k D(A_A))$. Applying $\mathrm{Hom}_\Lambda(-, S(i) \otimes_k D(A_A))$ to $0 \to \Omega(X) \to P_X \to X \to 0$ and using Lemma 3.2, we get the following exact sequence

$$0 \to (\mathrm{Coker}_i(X), D(A_A)) \to (\mathrm{Coker}_i(P_X), D(A_A)) \to (\mathrm{Coker}_i(\Omega(X)), D(A_A)) \to 0,$$

i.e., $0 \to \mathrm{Coker}_i(X) \to \mathrm{Coker}_i(P(X)) \to \mathrm{Coker}_i(\Omega(X)) \to 0$ is exact. Comparing it with $(*)$ we see $\mathrm{Ker}\, \delta_i(X) = 0$. ■

3.4. Replacing $D(A_A)$ in Proposition 3.4 by an arbitrary A-module T, we have

Proposition 3.5. *Let T be an A-module. Then* $\mathrm{Mon}(Q, {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T) \cap \mathrm{Mon}(Q, A)$.

Proof. We first prove that for each $i \in Q_0$ there holds the following equality

$$\begin{aligned} {}^{\perp}(S(i) \otimes_k T) \cap \mathrm{Mon}(Q, A) = \{X = (X_j, X_\alpha) \in \mathrm{Rep}(Q, A) \mid \mathrm{Coker}_i(X) \in {}^{\perp}T, \\ \delta_j(X) \text{ is injective for all } j \in Q_0\}. \end{aligned} \qquad (3.1)$$

Let $X \in \mathrm{Mon}(Q, A)$ with a projective resolution $\cdots \to P^1 \to P^0 \to X \to 0$. Since each P^i is in $\mathrm{Mon}(Q, A)$ (cf. Lemma 2.3) and $\mathrm{Mon}(Q, A)$ is closed under taking the kernels of epimorphisms (cf. Lemma 2.5), it follows from Lemma 2.4 that we have the exact sequence

$$\cdots \to \mathrm{Coker}_i(P^1) \to \mathrm{Coker}_i(P^0) \to \mathrm{Coker}_i(X) \to 0.$$

We claim it is a projective resolution of $\mathrm{Coker}_i(X)$. In fact, by (2.2) we have

$$\mathrm{Coker}_i(P(j) \otimes_k T) = \begin{cases} T, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases}$$

So $\mathrm{Coker}_i(kQ \otimes_k T) = T$ and $\mathrm{Coker}_i(kQ \otimes_k A) = A$. Thus $\mathrm{Coker}_i(P^j)$ is a projective A-module since $P^j \in \mathrm{add}(kQ \otimes_k A)$.

Applying $\mathrm{Hom}(-, S(i) \otimes_k T)$ to $\cdots \to P^1 \to P^0 \to X \to 0$, by Lemma 3.2 we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (X, S(i) \otimes_k T) & \longrightarrow & (P^0, S(i) \otimes_k T) & \longrightarrow & (P^1, S(i) \otimes_k T) & \longrightarrow & \cdots \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & (\mathrm{Coker}_i(X), T) & \longrightarrow & (\mathrm{Coker}_i(P^0), T) & \longrightarrow & (\mathrm{Coker}_i(P^1), T) & \longrightarrow & \cdots \end{array}$$

Note that $X \in {}^\perp(S(i) \otimes_k T)$ if and only if the upper row is exact, if and only if the lower one is exact, if and only if $\mathrm{Coker}_i(X) \in {}^\perp T$. This proves (3.1).

Now, assume that $X \in \mathrm{Mon}(Q, {}^\perp T)$. By definition and (3.1) we know $X \in {}^\perp(S(i) \otimes_k T) \cap \mathrm{Mon}(Q, A)$ for each $i \in Q_0$. By Lemma 3.3 we know $X \in {}^\perp(kQ \otimes_k T)$ and hence $X \in {}^\perp(kQ \otimes_k T) \cap \mathrm{Mon}(Q, A)$.

Conversely, assume that $X \in {}^\perp(kQ \otimes_k T) \cap \mathrm{Mon}(Q, A)$. By Lemma 3.3 $X \in {}^\perp(S(i) \otimes_k T) \cap \mathrm{Mon}(Q, A)$ for each $i \in Q_0$. To see $X \in \mathrm{Mon}(Q, {}^\perp T)$, by (3.1) it remains to prove $X_i \in {}^\perp T$ for each $i \in Q_0$. For each $i \in Q_0$, set $l_i = 0$ if i is a source, and $l_i = \max\{\ l(p)\ \mid p \text{ is a path with } e(p) = i\}$ if otherwise, where $l(p)$ is the length of p. We prove $X_i \in {}^\perp T$ by using induction on l_i. If $l_i = 0$, then i is a source and $X_i = \mathrm{Coker}_i(X) \in {}^\perp T$. Let $l_i \neq 0$. Then we have the exact sequence $0 \longrightarrow \bigoplus\limits_{\substack{\alpha \in Q_1 \\ e(\alpha) = i}} X_{s(\alpha)} \longrightarrow X_i \longrightarrow \mathrm{Coker}_i(X) \longrightarrow 0$ with $\mathrm{Coker}_i(X) \in {}^\perp T$. Since $l_{s(\alpha)} < l_i$ for $\alpha \in Q_1$ and $e(\alpha) = i$, by induction $X_{s(\alpha)} \in {}^\perp T$, and hence $X_i \in {}^\perp T$. This completes the proof. ∎

3.5. **Proof of Theorem 3.1.** By Proposition 3.5 it suffices to prove ${}^\perp(kQ \otimes_k T) \subseteq \mathrm{Mon}(Q,\ A)$. By Proposition 3.4 it suffices to prove ${}^\perp(kQ \otimes_k T) \subseteq {}^\perp(kQ \otimes_k D(A_A))$. Let $X \in {}^\perp(kQ \otimes_k T)$. By assumption we have an exact sequence $0 \longrightarrow kQ \otimes_k T_m \longrightarrow \cdots \longrightarrow kQ \otimes_k T_0 \longrightarrow kQ \otimes_k D(A_A) \longrightarrow 0$ with each $kQ \otimes_k T_j \in \mathrm{add}(kQ \otimes_k T)$. From this we see the assertion. ∎

3.6. Let $\mathcal{GP}(A)$ denote the category of the Gorenstein-projective A-modules. If A is Gorenstein (i.e., $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim} A_A < \infty$), then $\mathcal{GP}(A) = {}^\perp A$ ([EJ, Corollary 11.5.3]). Note that if A is Gorenstein then so is Λ. Taking $T = {}_A A$ in Theorem 3.1 we have

Corollary 3.6. *Let A be a Gorenstein algebra. Then $\mathcal{GP}(\Lambda) = \mathrm{Mon}(Q, \mathcal{GP}(A))$.*

4. Monomorphism categories and cotilting theory

4.1. The aim of this section is to prove the following

Theorem 4.1. *Let T be a cotilting A-module. Then $kQ \otimes_k T$ is a unique cotilting Λ-module, up to multiplicities of indecomposable direct summands, such that* $\mathrm{Mon}(Q,\ {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T)$.

For $Q = \bullet \to \cdots \to \bullet$ this result has been obtained in [Z, Theorem 3.1(ii)]. We stress that the proof in [Z] can not be generalized to the general case. Here we need to use Lemma 4.4 below, rather than a concrete construction in [Z, Lemma 3.7].

4.2. Recall that an A-module T is an r-cotilting module ([HR], [AR], [H], [Mi]) if the following conditions are satisfied:

(i) $\mathrm{inj.dim} T \le r$;

(ii) $\mathrm{Ext}^i_A(T, T) = 0$ for $i \ge 1$;

(iii) there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to D(A_A) \to 0$ with each $T_j \in \mathrm{add}(T)$.

For short, by $\mathbf{m}_i$ we denote the functor $P(i) \otimes_k - :\ A\text{-mod} \to \mathrm{Mon}(Q, A)$, and by $\mathbf{m}$ we denote the functor $kQ \otimes_k - : A\text{-mod} \to \mathrm{Mon}(Q, A)$. Then $\mathbf{m}(T) = \bigoplus\limits_{i \in Q_0} \mathbf{m}_i(T) = kQ \otimes_k T,\ \forall\ T \in A\text{-mod}$.

Lemma 4.2. ([LZ, Lemma 2.3]) *We have adjoint pair $(\mathbf{m}_i, F_i)$ for each $i \in Q_0$, where functor F_i is defined in 2.4.*

We also need the following fact.

Lemma 4.3. *Let $X = (X_j, X_\alpha) \in \Lambda$-mod and $T \in A$-mod. Then we have an isomorphism of abelian groups for each $i \in Q_0$, which is natural in both positions*

$$\mathrm{Ext}^s_\Lambda(\mathbf{m}_i(T), X) \cong \mathrm{Ext}^s_A(T, X_i), \quad \forall\ s \ge 0.$$

Proof. The proof is same as in [Z, Lemma 3.4] for $Q = \bullet \to \cdots \to \bullet$. For completeness we include a justification. Taking the i-th branch of an injective resolution $0 \to X \to I^0 \to I^1 \to \cdots$ of ${}_\Lambda X$, by Lemma 2.3 $0 \to X_i \to I^0_i \to I^1_i \to \cdots$ is an injective resolution of ${}_A X_i$. On the other hand by Lemma 4.2 we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_\Lambda(\mathbf{m}_i(T), X) & \longrightarrow & \mathrm{Hom}_\Lambda(\mathbf{m}_i(T), I^0) & \longrightarrow & \mathrm{Hom}_\Lambda(\mathbf{m}_i(T), I^1) & \longrightarrow & \cdots \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
0 & \longrightarrow & \mathrm{Hom}_A(T, X_i) & \longrightarrow & \mathrm{Hom}_A(T, I^0_i) & \longrightarrow & \mathrm{Hom}_A(T, I^1_i) & \longrightarrow & \cdots,
\end{array}$$

from this we see the assertion. ■

4.3. Let $\mathcal{X}$ be a full subcategory of A-mod. Following [AR] let $\widehat{\mathcal{X}}$ denote the full subcategory of A-mod consisting of those A-modules X such that there is an exact sequence $0 \to X_m \to X_{m-1} \to \cdots \to X_0 \to X \to 0$ with each $X_i \in \mathcal{X}$. Recall that $\mathcal{X}$ is *self-orthogonal* if $\mathrm{Ext}^s_A(M, N) = 0,\ \ \forall\ M,\ N \in \mathcal{X},\ \forall\ s \ge 1$. In this case $\widehat{\mathcal{X}} \subseteq \mathcal{X}^{\perp}$, where $\mathcal{X}^{\perp} = \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(M, X) = 0,\ \forall\ M \in \mathcal{X},\ \forall\ i \ge 1\}$.

The following fact is of independent interest. It is a key step in the proof of Theorem 4.1.

Lemma 4.4. *Let $\mathcal{X}$ be a self-orthogonal full subcategory of A-mod. Then*

(i) *$\widehat{\mathcal{X}}$ is closed under taking cokernels of monomorphisms.*

(ii) *$\widehat{\mathcal{X}}$ is closed under taking extensions.*

(iii) *If $\mathcal{X}$ is closed under taking kernels of epimorphisms, then so is $\widehat{\mathcal{X}}$.*

Proof. (i) Let $0 \to X \xrightarrow{f} Y \to Z \to 0$ be an exact sequence with $X, Y \in \widehat{\mathcal{X}}$. By definition there exist exact sequences $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \xrightarrow{c_0} X \to 0$, and $0 \to Y_n \to Y_{n-1} \to \cdots \to Y_0 \xrightarrow{d_0} Y \to 0$ with $X_i,\ Y_i \in \mathcal{X} \cup \{0\},\ 0 \le i \le n$. Since $\mathcal{X}$ is self-orthogonal, $f : X \to Y$ induces a chain map $f^\bullet : X^\bullet \to Y^\bullet$, where $X^\bullet$ is the complex $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to 0$, and similarly for $Y^\bullet$. Consider the following commutative diagram in the bounded derived category $D^b(A)$, with two rows being distinguished triangles

$$\begin{tikzcd} X^\bullet \arrow[r, "f^\bullet"] \arrow[d, "c_0"] & Y^\bullet \arrow[r] \arrow[d, "d_0"] & \mathrm{Con}(f^\bullet) \arrow[r] \arrow[d, dotted] & X^\bullet[1] \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

(note that the lower row is also a distinguished triangle since $0 \to X \xrightarrow{f} Y \to Z \to 0$ is exact), where $\mathrm{Con}(f^\bullet)$ is the mapping cone $0 \to X_n \to X_{n-1} \oplus Y_n \to \cdots \to X_0 \oplus Y_1 \xrightarrow{\partial} Y_0 \to 0$. Since c_0 and d_0 are isomorphisms in $D^b(A)$, we have $Z \cong \mathrm{Con}(f^\bullet)$ in $D^b(A)$. It follows that the i-th cohomology group of $\mathrm{Con}(f^\bullet)$ is isomorphic to the i-th cohomology group of the stalk complex Z for each $i \in \mathbb{Z}$. In particular $\mathrm{Con}(f^\bullet)$ is exact except at the 0-th position, and $Y_0/\operatorname{Im}\partial \cong Z$. Thus

$$0 \to X_n \to X_{n-1} \oplus Y_n \to \cdots \to X_0 \oplus Y_1 \xrightarrow{\partial} Y_0 \to Z \to 0$$

is exact. This proves $Z \in \widehat{\mathcal{X}}$.

(iii) can be similarly proved, and (ii) can be proved by a version of Horse-shoe Lemma. We omit the details. (Only (i) will be needed in the proof of Theorem 4.1.) ■

Lemma 4.5. *Let T be an r-cotilting A-module. Then $kQ \otimes_k T$ is an $(r+1)$-cotilting Λ-module with $\mathrm{End}_\Lambda(kQ \otimes_k T) \cong (kQ \otimes_k \mathrm{End}_A(T))^{op}$.*

Proof. Let $0 \to T \to I_0 \to \cdots \to I_r \to 0$ be a minimal injective resolution of T. Then we have the exact sequence $0 \to kQ \otimes_k T \to kQ \otimes_k I_0 \to \cdots \to kQ \otimes_k I_r \to 0$. By Lemma 2.3 $\mathrm{inj.dim}(kQ \otimes_k I_j) \le 1,\ \ 0 \le j \le r$, it follows that $\mathrm{inj.dim}(kQ \otimes_k T) \le r+1$.

Since the branch $(kQ \otimes_k T)_i$ is a direct sum of copies of T, by Lemma 4.3 we have

$$\begin{aligned} \mathrm{Ext}^s_\Lambda(kQ \otimes_k T, kQ \otimes_k T) &= \bigoplus_{i \in Q_0} \mathrm{Ext}^s_\Lambda(\mathbf{m}_i(T), kQ \otimes_k T) \\ &\cong \bigoplus_{i \in Q_0} \mathrm{Ext}^s_A(T, (kQ \otimes_k T)_i) = 0, \quad \forall\ s \ge 1. \end{aligned}$$

Now, put $\mathcal{X} = \mathrm{add}(kQ \otimes_k T)$. To see that $kQ \otimes_k T$ is a cotilting Λ-module, it remains to claim $D(\Lambda_\Lambda) \in \widehat{\mathcal{X}}$, i.e., $D(kQ_{kQ}) \otimes_k D(A_A) \in \widehat{\mathcal{X}}$. In fact, since $\mathrm{proj.dim}\ D(kQ_{kQ}) = 1$, we have an

exact sequence $0 \to P_1 \to P_0 \to D(kQ_{kQ}) \to 0$ with P_0, P_1 being projective kQ-modules. So we have the exact sequence $0 \to P_1 \otimes_k D(A_A) \to P_0 \otimes_k D(A_A) \to D(kQ_{kQ}) \otimes_k D(A_A) \to 0$. Since T is a cotilting A-module, we have an exact sequence $0 \to T_m \to \cdots \to T_0 \to D(A_A) \to 0$ with each $T_j \in \mathrm{add}(T)$. So we have the exact sequence $0 \to P_i \otimes_k T_m \to \cdots \to P_i \otimes_k T_0 \to P_i \otimes_k D(A_A) \to 0$, where $i = 0, 1$, with each $P_i \otimes T_j \in \mathrm{add}(kQ \otimes_k T)$. Thus $P_0 \otimes_k D(A_A) \in \widehat{\mathcal{X}}$ and $P_1 \otimes_k D(A_A) \in \widehat{\mathcal{X}}$. By Lemma 4.4(i) we have $D(kQ_{kQ}) \otimes_k D(A_A) \in \widehat{\mathcal{X}}$.

Finally, by Lemma 4.2 we have

$$\mathrm{Hom}_\Lambda(\mathbf{m}_i(T), \mathbf{m}_j(T)) \cong \mathrm{Hom}_A(T, (\mathbf{m}_j(T))_i) = (\mathrm{End}_A(T))^{m_{ji}},$$

where m_{ji} is the number of paths of Q from j to i. Thus one can easily see that there is an algebra isomorphism

$$\mathrm{End}_\Lambda(kQ \otimes_k T) \cong \bigoplus_{i,j \in Q_0} \mathrm{Hom}_\Lambda(\mathbf{m}_i(T), \mathbf{m}_j(T)) \cong (kQ \otimes_k \mathrm{End}_A(T))^{op}.$$

(In fact, if we label the vertices of Q as $1, \cdots, n$, such that if there is an arrow from j to i then $j > i$. Then

$$kQ \cong \begin{pmatrix} k & k^{m_{21}} & k^{m_{31}} & \cdots & k^{m_{n1}} \\ 0 & k & k^{m_{32}} & \cdots & k^{m_{n2}} \\ 0 & 0 & k & \cdots & k^{m_{n3}} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix}_{n \times n},$$

and hence

$$kQ \otimes_k \mathrm{End}_A(T) \cong \begin{pmatrix} \mathrm{End}_A(T) & \mathrm{End}_A(T)^{m_{21}} & \mathrm{End}_A(T)^{m_{31}} & \cdots & \mathrm{End}_A(T)^{m_{n1}} \\ 0 & \mathrm{End}_A(T) & \mathrm{End}_A(T)^{m_{32}} & \cdots & \mathrm{End}_A(T)^{m_{n2}} \\ 0 & 0 & \mathrm{End}_A(T) & \cdots & \mathrm{End}_A(T)^{m_{n3}} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{End}_A(T) \end{pmatrix}_{n \times n}.)$$

This completes the proof. ■

4.4. **Proof of Theorem 4.1.** By Lemma 4.5 $kQ \otimes_k T$ is a cotilting Λ-module, and by Theorem 3.1 $\mathrm{Mon}(Q, {}^\perp T) = {}^\perp(kQ \otimes_k T)$.

If L is another cotilting Λ-module such that ${}^\perp L = \mathrm{Mon}(Q, {}^\perp T) = {}^\perp(kQ \otimes_k T)$, then

$$\mathrm{Ext}^s_\Lambda((kQ \otimes_k T) \oplus L, (kQ \otimes_k T) \oplus L) = 0, \quad \forall\ s \geq 1,$$

so $(kQ \otimes_k T) \oplus L$ is also a cotilting Λ-module. By [H] the number of pairwise non-isomorphic direct summands of $(kQ \otimes_k T) \oplus L$ is equal to the one of $kQ \otimes_k T$, from which the proof is completed. ■

5. **Contravariantly finiteness of monomorphism categories**

5.1. Let $\mathcal{X}$ be a full subcategory of A-mod and $M \in A$-mod. Recall from [AR] that *a right $\mathcal{X}$-approximation of* M is an A-map $f : X \longrightarrow M$ with $X \in \mathcal{X}$, such that the induced homomorphism $\mathrm{Hom}_A(X', X) \longrightarrow \mathrm{Hom}_A(X', M)$ is surjective for $X' \in \mathcal{X}$. If every A-module M admits a right $\mathcal{X}$-approximation, then $\mathcal{X}$ is *contravariantly finite in* A-mod. Dually one has the concept of *a covariantly finite subcategory.* If $\mathcal{X}$ is both contravariantly and covariantly finite, then $\mathcal{X}$ is *functorially finite in* A-mod. Due to H. Krause and Ø. Solberg [KS, Corollary 0.3], a resolving

contravariantly finite subcategory is functorially finite, and a coresolving covariantly finite subcategory is functorially finite. Due to M. Auslander and S. O. Smalø [AS, Theorem 2.4], a functorially finite subcategory which is closed under taking extensions has Auslander-Reiten sequences.

5.2. Auslander-Reiten [AR, Theorem 5.5(a)] claim that $\mathcal{X}$ is resolving and contravariantly finite with $\widehat{\mathcal{X}} = A$-mod if and only if $\mathcal{X} = {}^{\perp}T$ for some cotilting A-module T, where $\widehat{\mathcal{X}}$ is defined in 4.3.

As an application of Theorem 4.1 and [AR, Theorem 5.5(a)], we have

Theorem 5.1. *Let $\mathcal{X}$ be a full subcategory of A-mod. Then* $\operatorname{Mon}(\mathrm{Q}, \mathcal{X})$ *is a resolving contravariantly finite subcategory in Λ-mod with* $\widehat{\operatorname{Mon}(Q, \mathcal{X})} = \Lambda$*-mod if and only if $\mathcal{X}$ is a resolving contravariantly finite subcategory in A-mod with $\widehat{\mathcal{X}} = A$-mod.*

In particular, $\operatorname{Mon}(Q, A)$ *is functorially finite in* $\operatorname{Rep}(Q, A)$*, and* $\operatorname{Mon}(Q, A)$ *has Auslander-Reiten sequences.*

Proof. If $\mathcal{X}$ is resolving and contravariantly finite with $\widehat{\mathcal{X}} = A$-mod, then by [AR, Theorem 5.5(a)] there is a cotilting module T such that $\mathcal{X} = {}^{\perp}T$. By Theorem 4.1 $kQ \otimes_k T$ is a cotilting Λ-module and $\operatorname{Mon}(Q, \mathcal{X}) = \operatorname{Mon}(Q, {}^{\perp}T) = {}^{\perp}(kQ \otimes_k T)$, again by [AR, Theorem 5.5(a)] we know that $\operatorname{Mon}(Q, \mathcal{X})$ is resolving and contravariantly finite with $\widehat{\operatorname{Mon}(Q, \mathcal{X})} = \Lambda$-mod.

Conversely, assume that $\operatorname{Mon}(Q, \mathcal{X})$ is resolving and contravariantly finite with $\widehat{\operatorname{Mon}(Q, \mathcal{X})} = \Lambda$-mod. By Lemma 2.5 $\mathcal{X}$ is resolving. To see that $\mathcal{X}$ is contravariantly finite, we take a sink in Q_0, say vertex 1, and consider functor $\mathbf{m}_1 : A\text{-mod} \to \operatorname{Mon}(Q, A)$ (cf. 4.2). For $M \in A$-mod, since 1 is a sink, $\mathbf{m}_1(M)$ has only one non-zero branch and its 1-st branch is just M. Let $f : X \longrightarrow \mathbf{m}_1(M)$ be a right $\operatorname{Mon}(Q, \mathcal{X})$-approximation. Then $f_1 : X_1 \longrightarrow M$ is a right $\mathcal{X}$-approximation (one can easily see this, for example, by Lemma 4.2. We omit the details). By the same argument we see $\widehat{\mathcal{X}} = A$-mod since $\widehat{\operatorname{Mon}(Q, \mathcal{X})} = \Lambda$-mod. This completes the proof. ■

6. Finiteness of monomorphism categories

As an application of Theorem 4.1 and Auslander's classical idea [Au, Chapter III], we describe the monomorphism categories which are of finite type.

6.1. An additive full subcategory $\mathcal{X}$ of A-mod, which is closed under direct summands, *is of finite type* if there are only finitely many isomorphism class of indecomposable A-modules in $\mathcal{X}$.

An A-module M is *an A-generator* if each projective A-module is in $\operatorname{add}(M)$. A Λ-generator M is *a generator and relative cogenerator of* $\operatorname{Mon}(Q, A)$ if $M \in \operatorname{Mon}(Q, A)$ and $kQ \otimes_k D(A_A) \in \operatorname{add}(M)$.

Theorem 6.1. $\operatorname{Mon}(Q, A)$ *if of finite type if and only if there is a generator and relative cogenerator M of* $\operatorname{Mon}(Q, A)$ *such that* $\operatorname{gl.dim} \operatorname{End}_{\Lambda}(M) \le 2$.

6.2. Let M be a Λ-module. For an arbitrary Λ-module X, denote by $\Omega_M(X)$ the kernel of a minimal right $\operatorname{add}(M)$-approximation $M' \longrightarrow X$ of X. Define $\Omega^0_M(X) = X$, and $\Omega^i_M(X) = \Omega_M(\Omega^{i-1}_M(X))$ for $i \ge 1$. Define $\operatorname{rel.dim}_M X$ to be the minimal non-negative integer d such that $\Omega^d_M(X) \in \operatorname{add}(M)$, or ∞ if otherwise. The following fact is well known.

Lemma 6.2. (M. Auslander) *Let M be an A-module with $\Gamma = (\mathrm{End}_A(M))^{op}$. Then for each A-module X we have* $\mathrm{proj.dim}_\Gamma \mathrm{Hom}_A(M, X) \le \mathrm{rel.dim}_M X$. *Furthermore, if M is a generator, then equality holds.*

For an A-module T, denote by $\mathcal{X}_T$ the full subcategory of A-mod given by

$\{X \mid \exists$ an exact sequence $0 \to X \to T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots,$ with $T_i \in \mathrm{add}(T),\ \mathrm{Ker}\, d_i \in {}^\perp T, \forall\ i \ge 0\}$.

Not that $\mathcal{X}_T \subseteq {}^\perp T$, and $\mathcal{X}_T = {}^\perp T$ if T is a cotilting module ([AR, Theorem 5.4(b)]).

Lemma 6.3. *Let M be an A-generator with $\Gamma = (\mathrm{End}_A(M))^{op}$, and $T \in \mathrm{add}(M)$. Then for each A-module $X \in \mathcal{X}_T$ and $X \notin \mathrm{add}(T)$, there is a Γ-module Y such that* $\mathrm{proj.dim}_\Gamma Y = 2 + \mathrm{proj.dim}_\Gamma \mathrm{Hom}_A(M, X)$.

Proof. This is well-known. For completeness we include a proof. By $X \in \mathcal{X}_T$ there is an exact sequence $0 \to X \overset{u}{\to} T_0 \overset{v}{\to} T_1$ with $T_0,\ T_1 \in \mathrm{add}(T) \subseteq \mathrm{add}(M)$. This yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, X) \xrightarrow{u_*} \mathrm{Hom}_A(M, T_0) \xrightarrow{v_*} \mathrm{Hom}_A(M, T_1) \longrightarrow \mathrm{Coker}\, v_* \longrightarrow 0.$$

Note the image of v_* is not projective (otherwise, u_* splits, then we have an A-map $u' : T_0 \longrightarrow X$ such that $\mathrm{Hom}_A(M, u'u) = \mathrm{Hom}_A(M, \mathrm{Id}_X)$. Since M is an A-generator, we then get $u'u = \mathrm{Id}_X$. This contradicts with $X \in \mathrm{add}(T)$). Putting $Y = \mathrm{Coker}\, v_*$, we have $\mathrm{proj.dim}_\Gamma Y = 2 + \mathrm{proj.dim}_\Gamma \mathrm{Hom}_A(M, X)$. ■

6.3. **Proof of Theorem 6.1.** This is same as [Z, 5.3]. For completeness we include a proof.

Assume that $\mathrm{Mon}(Q, A)$ is of finite type. Then there is a Λ-module M such that $\mathrm{Mon}(Q, A) = \mathrm{add}(M)$. Since $kQ \otimes_k D(A_A) \in \mathrm{Mon}(Q, A)$, and $\mathrm{Mon}(Q, A)$ contains all the projective Λ-modules, by definition M is a generator and relative cogenerator of $\mathrm{Mon}(Q, A)$. Put $\Gamma = (\mathrm{End}_A(M))^{op}$. For every Γ-module Y, take a projective presentation $\mathrm{Hom}_\Lambda(M, M_1) \xrightarrow{f_*} \mathrm{Hom}_\Lambda(M, M_0) \to Y \to 0$ of Y, where $M_1, M_0 \in \mathrm{add}(M)$, and $f : M_1 \to M_0$ is a Λ-map. Since $\mathrm{inj.dim}(kQ \otimes D(A_A)) = 1$ (Lemma 2.3) and $M_1 \in \mathrm{Mon}(Q, A) = {}^\perp(kQ \otimes D(A_A))$ (Proposition 3.4), it follows that $\mathrm{Ker}\, f \in {}^\perp(kQ \otimes D(A_A)) = \mathrm{add}(M)$. Thus

$$0 \to \mathrm{Hom}_\Lambda(M, \mathrm{Ker}\, f) \to \mathrm{Hom}_\Lambda(M, M_1) \to \mathrm{Hom}_\Lambda(M, M_0) \to Y \to 0$$

is a projective resolution of Γ-module of Y, i.e., $\mathrm{proj.dim}_\Gamma Y \le 2$. This proves $\mathrm{gl.dim}_\Gamma Y \le 2$, and hence $\mathrm{gl.dim}\, \mathrm{End}_\Lambda(M) = \mathrm{gl.dim}\Gamma \le 2$.

Conversely, assume that there is a generator and relative cogenerator M of $\mathrm{Mon}(Q, A)$ such that $\mathrm{gl.dim}\, \mathrm{End}_\Lambda(M) \le 2$. Put $\Gamma = (\mathrm{End}_A(M))^{op}$. Then $\mathrm{gl.dim}\Gamma \le 2$. We claim that $\mathrm{add}(M) = {}^\perp(kQ \otimes D(A_A))$, and hence by Proposition 3.4 $\mathrm{Mon}(Q, A) = {}^\perp(kQ \otimes D(A_A)) = \mathrm{add}(M)$, i.e., $\mathrm{Mon}(Q, A)$ is of finite type. In fact, since $M \in \mathrm{Mon}(Q, A) = {}^\perp(kQ \otimes D(A_A))$, it follows that $\mathrm{add}(M) \subseteq {}^\perp(kQ \otimes D(A_A))$. On the other hand, let $X \in {}^\perp(kQ \otimes D(A_A))$. By Theorem 4.1 $kQ \otimes D(A_A)$ is a cotilting Λ-module, and hence ${}^\perp(kQ \otimes D(A_A)) = \mathcal{X}_{kQ \otimes D(A_A)}$, by [AR, Theorem 5.4(b)]. We divide into two cases. If $X \in \mathrm{add}(kQ \otimes D(A_A))$, then $X \in \mathrm{add}(M)$ since by assumption $kQ \otimes D(A_A) \in \mathrm{add}(M)$. If $X \notin \mathrm{add}(kQ \otimes D(A_A))$, then by Lemma 6.3 there is a Γ-module Y such that $\mathrm{proj.dim}_\Gamma Y = 2 + \mathrm{proj.dim}_\Gamma \mathrm{Hom}_\Lambda(M, X)$. Now by Lemma 6.2 we have

$$\mathrm{rel.dim}_M X = \mathrm{proj.dim}_\Gamma \mathrm{Hom}_\Lambda(M, X) = \mathrm{proj.dim}_\Gamma Y - 2 \le 0,$$

this means $X \in \mathrm{add}(M)$. This proves the claim and hence completes the proof. ■

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